Heuristic (Informed) **Search**

(Where we try to choose smartly)

Russell and Norvig:
Chap. 3, Sect. 3.5

Slides from Jean-Claude Latombe at Stanford University
(used with permission)

Search Algorithm #2

1. INSERT(N_0,FRONTIER)
2. Repeat:
   a. If EMPTY?(FRONTIER) then return failure
   b. N = POP(FRONTIER)
   c. s = STATE(N)
   d. If GOAL?(s) then return path or goal state
   e. For every state s' in SUCCESSORS(s)
      i. Create a new node N' as a child of N
      ii. INSERT(N',FRONTIER)

Expansion of N

Recall that the ordering of nodes in **FRONTIER** defines the search strategy

Are We Smart Yet?

- So far we've been "blundering about in the dark" - Let's try to be smarter!
- **Informed strategies** could find solutions more efficiently than uninformed ones
- We'll consider a new kind of search called **Best-First Search**, which chooses nodes for expansion based on an evaluation function

Best-First Search

- It exploits **state description** to estimate how “good” each search node is
- An **evaluation function** f maps each node N of the search tree to a real number:

  $f(N) \geq 0$

  [Traditionally, f(N) is an estimated cost; so, the smaller f(N), the more promising N]
- **Best-first search** sorts the FRONTIER in increasing f

  [Arbitrary order is assumed among nodes with equal f]

Best-First Search

- It exploits **state description** to estimate how "good" each search node is
- An **evaluatio** ode N of the search

  f(N) ≥ 0

  [Traditionally, f(N) is an estimated cost; so, the smaller f(N), the more promising N]
- **Best-first search** sorts the FRONTIER in increasing f

  [Arbitrary order is assumed among nodes with equal f]

"Best" does not refer to the quality of the generated path
Best-first search does not generate optimal paths in general

How to construct f?

- Typically, f(N) estimates:
  - either the cost of a solution path through N
    - Then f(N) = g(N) + h(N), where
      - g(N) is the cost of the path from the initial node to N
      - h(N) is an estimate of the cost of a path from N to a goal node
  - or the cost of a path from N to a goal node
    - Then f(N) = h(N) ▸ Greedy best-search
- But there are no limitations on f.
  Any function of your choice is acceptable.
  But will it help the search algorithm?

How to construct f?

- Typically, f(N) estimates:
  - either the cost of a solution path through N
    - Then f(N) = g(N) + h(N), where
      - g(N) is the cost of the path from the initial node to N
      - h(N) is an estimate of the cost of a path from N to a goal node
  - or the cost of a path from N to
    - Then f(N) = h(N)
- But there are no limitations on f.
  Any function of your choice is acceptable.
  But will it help the search algorithm?

Heuristic function

Heuristic Function

- The **heuristic function** $h(N) \geq 0$ estimates the cost to go from STATE(N) to a goal state

  Its value is independent of the current search tree; it depends only on STATE(N) and the goal test GOAL?

- Example:

5		8
4	2	1
7	3	6

STATE(N)

1	2	3
4	5	6
7	8	

Goal state

$h_1(N)$ = number of misplaced numbered tiles = 6

[Why is it an estimate of the distance to the goal?]

Other Examples

5		8
4	2	1
7	3	6

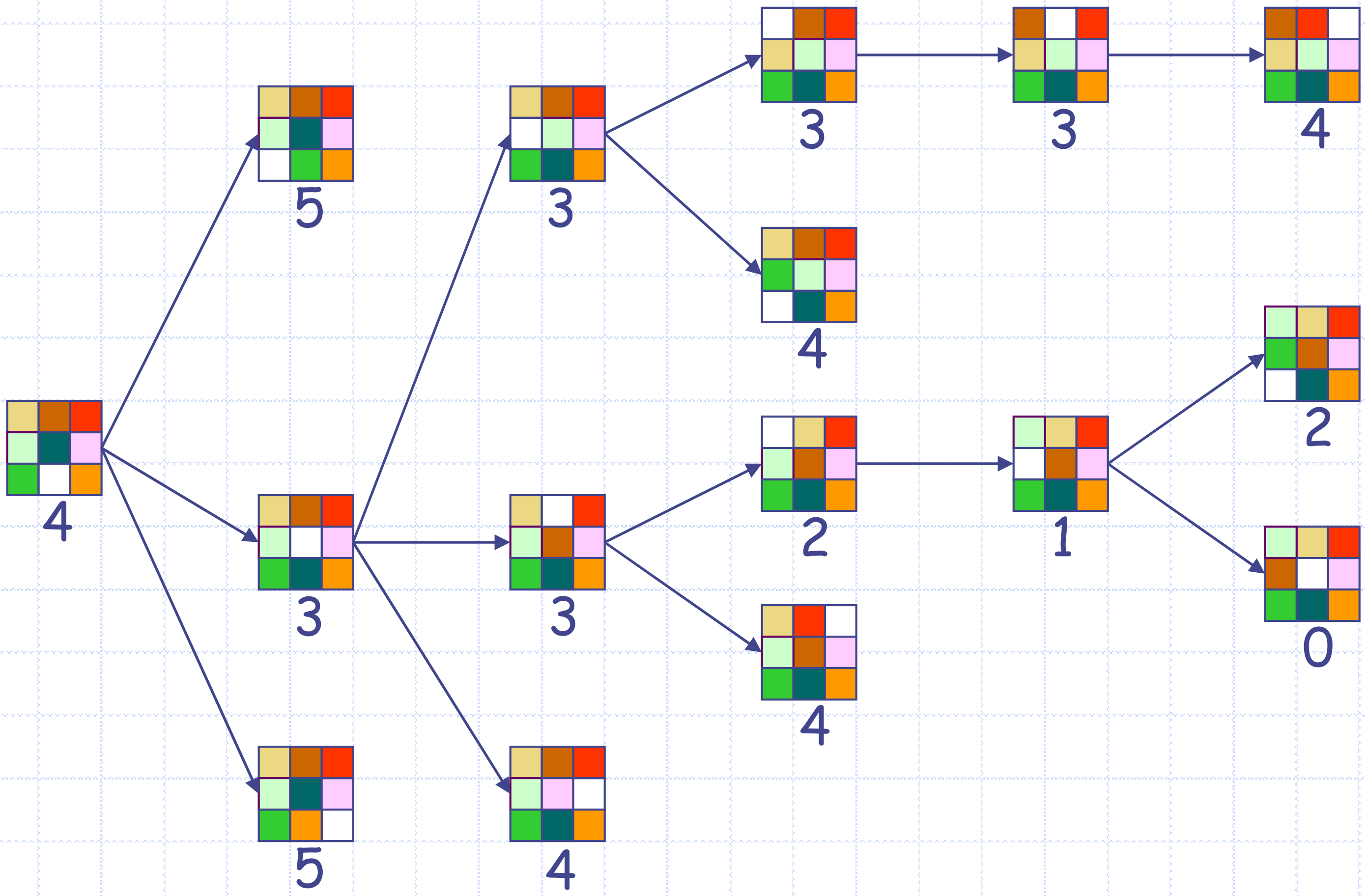
STATE(N)

1	2	3
4	5	6
7	8	

Goal state

- $h_1(N)$ = number of misplaced numbered tiles = 6
- $h_2(N)$ = sum of the (Manhattan) distance of every numbered tile to its goal position
  $= 2 + 3 + 0 + 1 + 3 + 0 + 3 + 1 = 13$
- $h_3(N)$ = sum of permutation inversions
  $= n_5 + n_8 + n_4 + n_2 + n_1 + n_7 + n_3 + n_6$
  $= 4 + 6 + 3 + 1 + 0 + 2 + 0 + 0$
  $= 16$

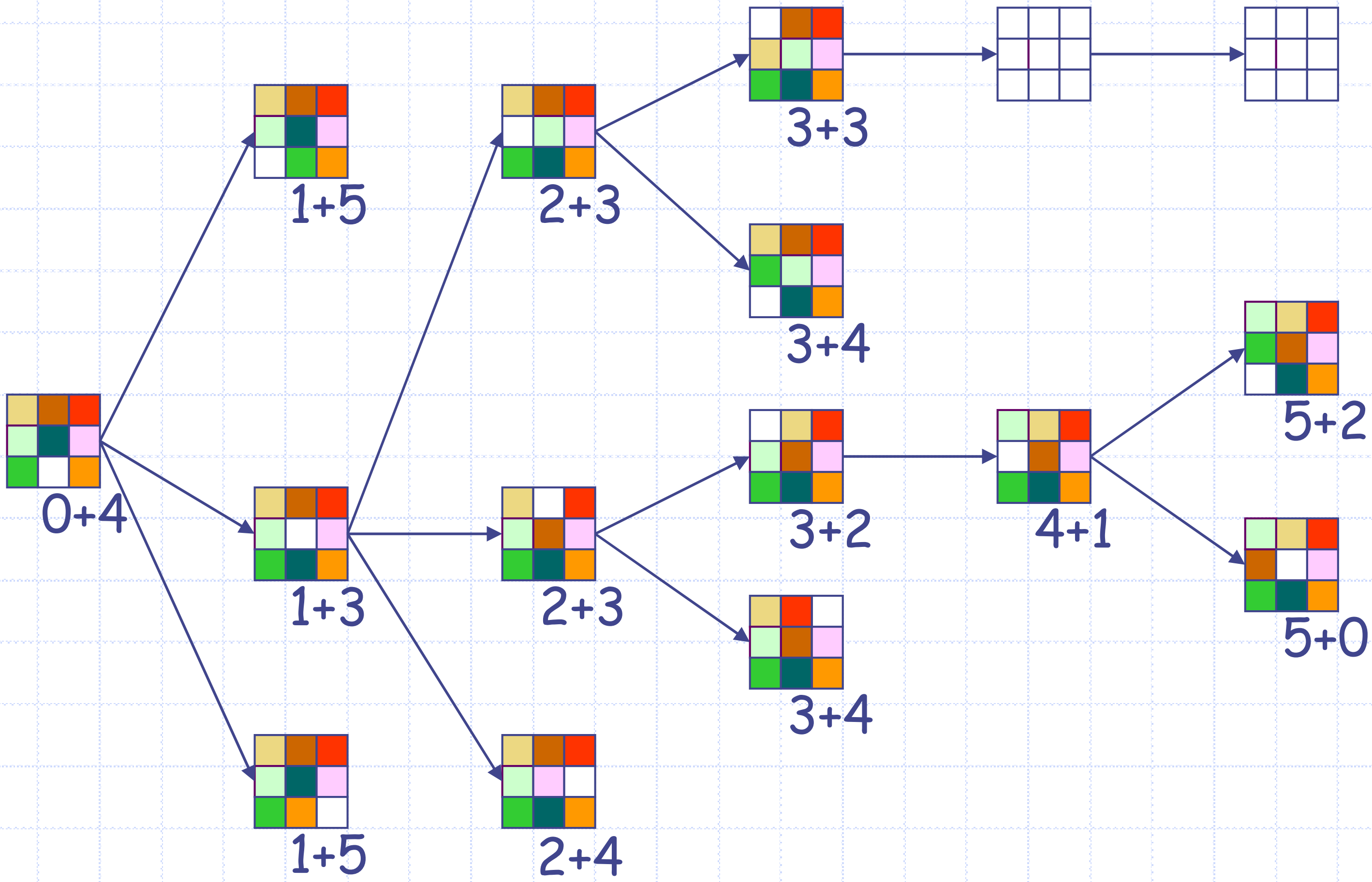
8-Puzzle f(N) = h(N) = number of misplaced tiles



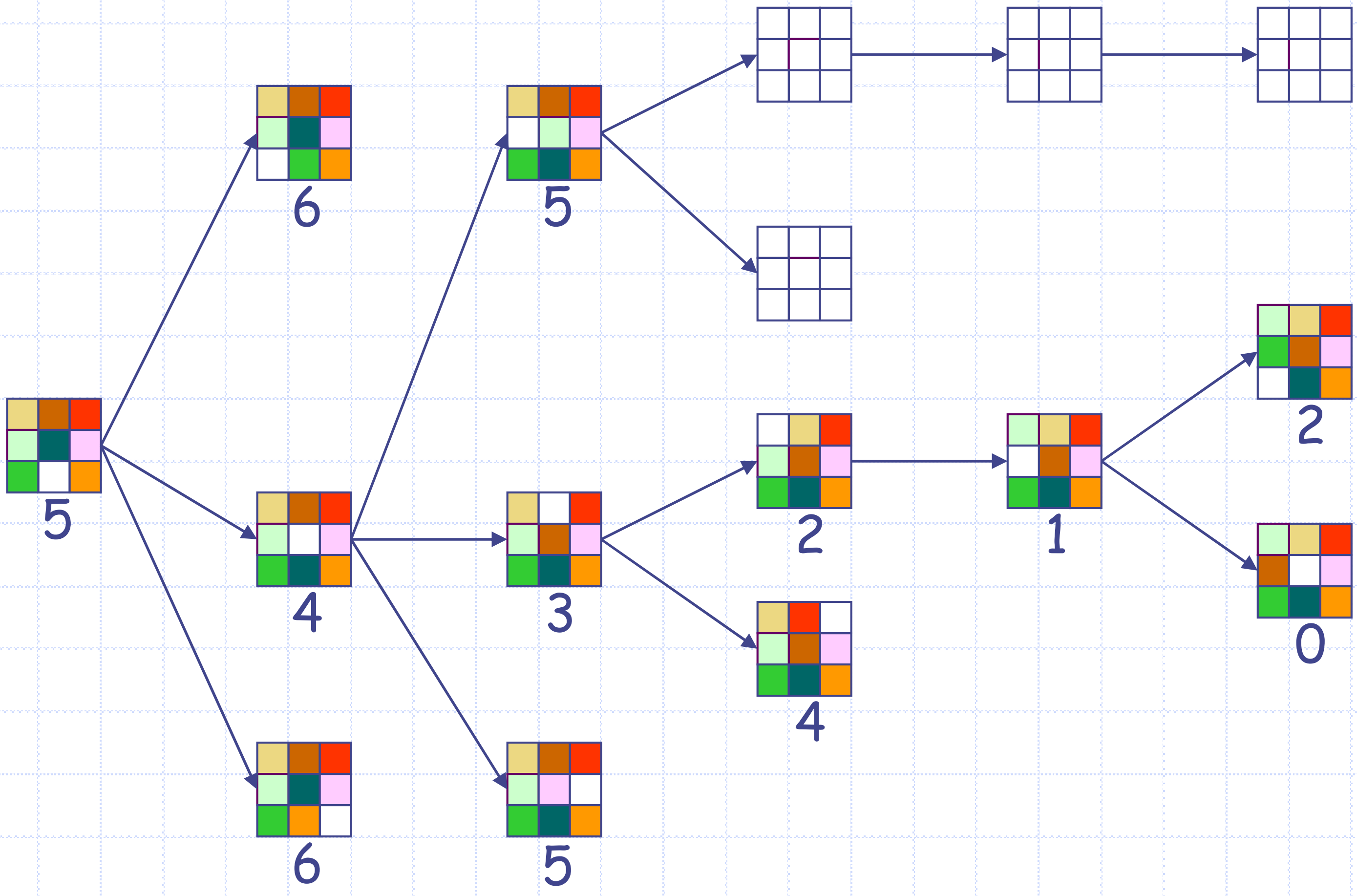
The white tile is the empty tile

8-Puzzle

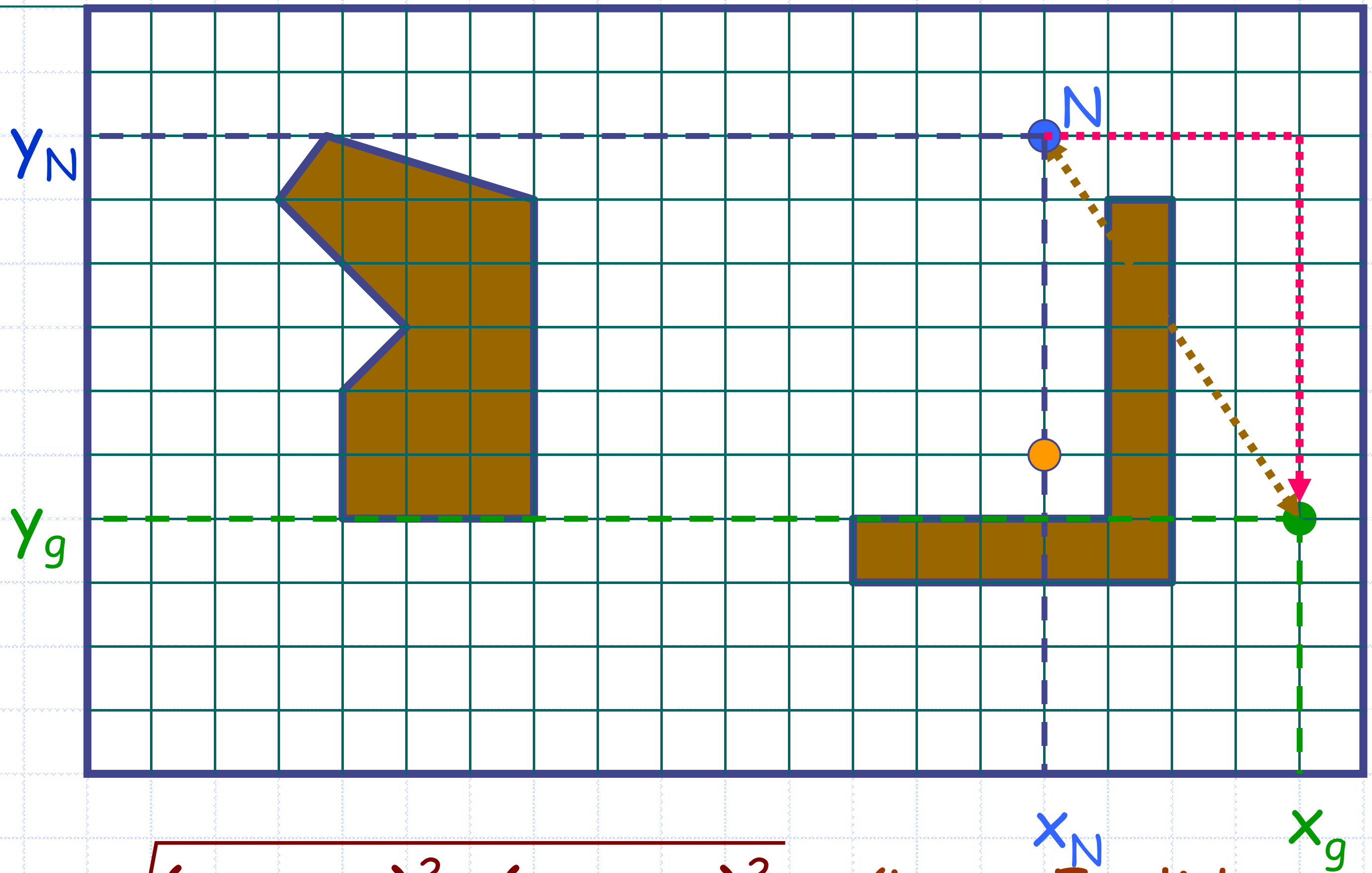
$f(N) = g(N) + h(N)$
with $h(N)$ = number of misplaced tiles



8-Puzzle $f(N) = h(N) = \Sigma$ distances of tiles to goals



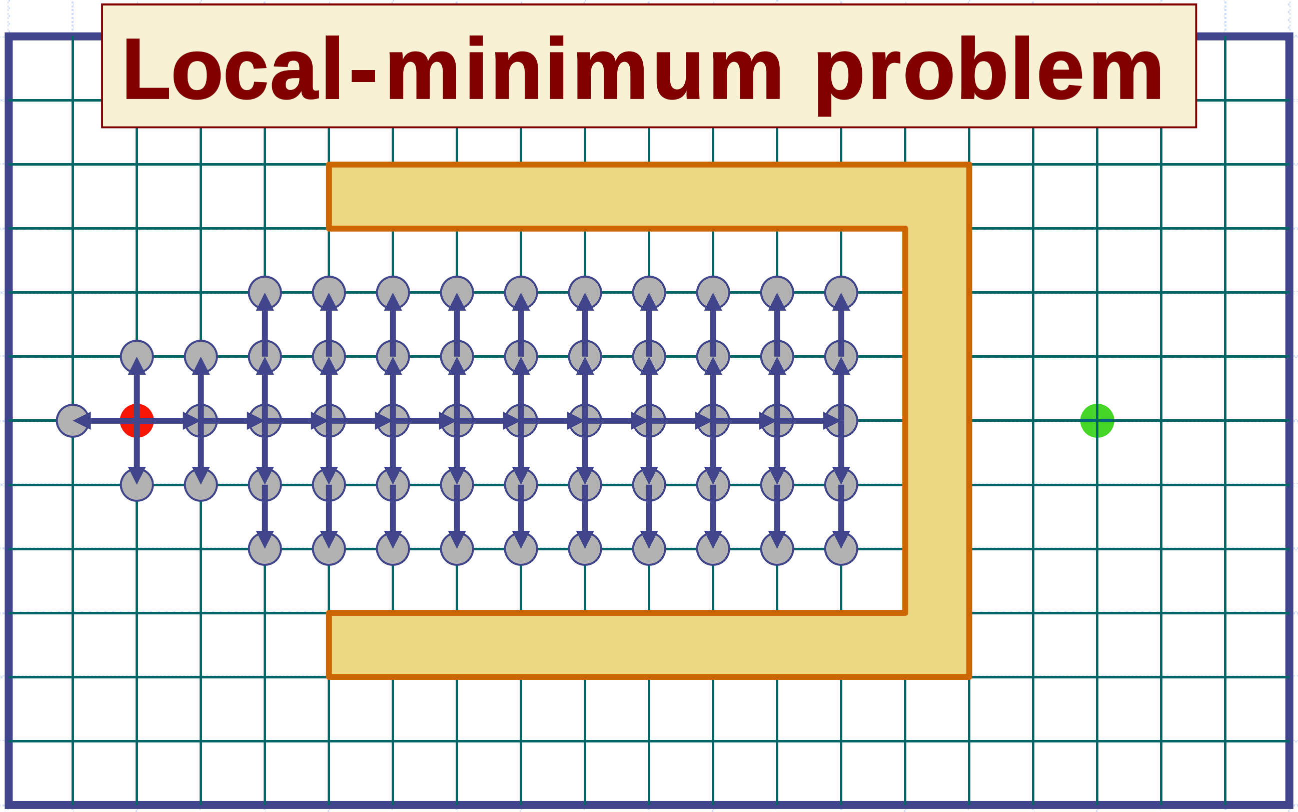
Robot Navigation



$$h_1(N) = \sqrt{(x_N - x_g)^2 + (y_N - y_g)^2}$$ (L_2 or Euclidean distance)

$$h_2(N) = |x_N - x_g| + |y_N - y_g|$$ (L_1 or Manhattan distance)

Best-First $\not\to$ Efficiency



f(N) = h(N) = straight distance to the goal

How Good is Best-First?

- If the state space is **infinite**, in general the search is **not complete**
- If the state space is **finite** and we do not discard nodes that **revisit states**, in general the search is **not complete**
- If the state space is **finite** and we **discard nodes** that revisit states, the search is **complete**, but in **general is not optimal**

Admissible Heuristic

- Let h*(N) be the cost of the optimal path from N to a goal node

- The heuristic function h(N) is **admissible** if:

$$0 \leq h(N) \leq h^*(N)$$

- An admissible heuristic function is always optimistic !

Admissible Heuristic

- Let h*(N) be the cost of the optimal path from N to a goal node

- The heuristic function h(N) is **admissible** if:

$$0 \le h(N) \le h^*(N)$$

G is a goal node
▸ h(G) = 0

- An admissible heuristic function is always optimistic !

8-Puzzle Heuristics

5		8
4	2	1
7	3	6

STATE(N)

1	2	3
4	5	6
7	8	

Goal state

- $h_1(N)$ = number of misplaced tiles = 6
  is ???
- $h_2(N)$ = sum of the (Manhattan) distances of every tile to its goal position
  = 2 + 3 + 0 + 1 + 3 + 0 + 3 + 1 = 13
  is ???
- $h_3(N)$ = sum of permutation inversions
  = 4 + 6 + 3 + 1 + 0 + 2 + 0 + 0 = 16
  is ???

8-Puzzle Heuristics

5		8
4	2	1
7	3	6

STATE(N)

1	2	3
4	5	6
7	8	

Goal state

- $h_1(N)$ = number of misplaced tiles = 6
  is ADMISSIBLE
- $h_2(N)$ = sum of the (Manhattan) distances of every tile to its goal position
  = 2 + 3 + 0 + 1 + 3 + 0 + 3 + 1 = 13
  is ???
- $h_3(N)$ = sum of permutation inversions
  = 4 + 6 + 3 + 1 + 0 + 2 + 0 + 0 = 16
  is ???

8-Puzzle Heuristics

5		8
4	2	1
7	3	6

STATE(N)

1	2	3
4	5	6
7	8	

Goal state

- $h_1(N)$ = number of misplaced tiles = 6
  is ADMISSIBLE
- $h_2(N)$ = sum of the (Manhattan) distances of every tile to its goal position
  $= 2 + 3 + 0 + 1 + 3 + 0 + 3 + 1 = 13$
  is ADMISSIBLE
- $h_3(N)$ = sum of permutation inversions
  $= 4 + 6 + 3 + 1 + 0 + 2 + 0 + 0 = 16$
  is ???

8-Puzzle Heuristics

5		8
4	2	1
7	3	6

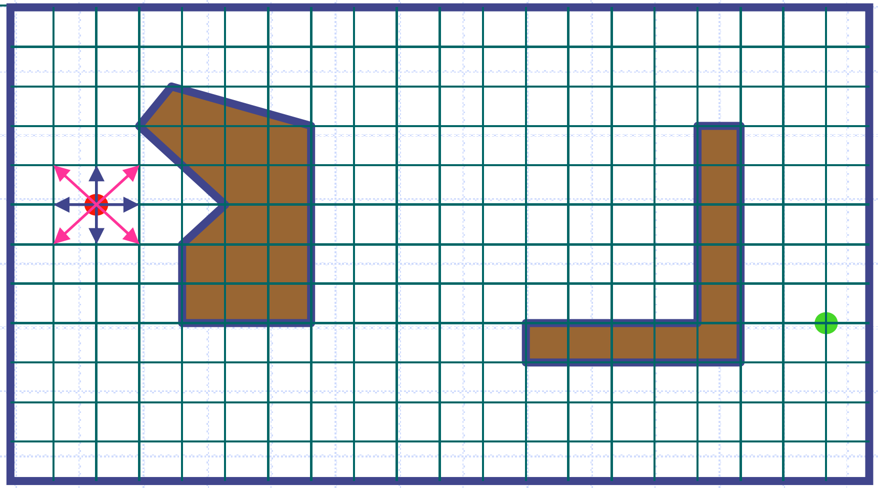
STATE(N)

1	2	3
4	5	6
7	8	

Goal state

- $h_1(N)$ = number of misplaced tiles = 6
  is ADMISSIBLE
- $h_2(N)$ = sum of the (Manhattan) distances of every tile to its goal position
  = 2 + 3 + 0 + 1 + 3 + 0 + 3 + 1 = 13
  is ADMISSIBLE
- $h_3(N)$ = sum of permutation inversions
  = 4 + 6 + 3 + 1 + 0 + 2 + 0 + 0 = 16
  is NOT ADMISSIBLE

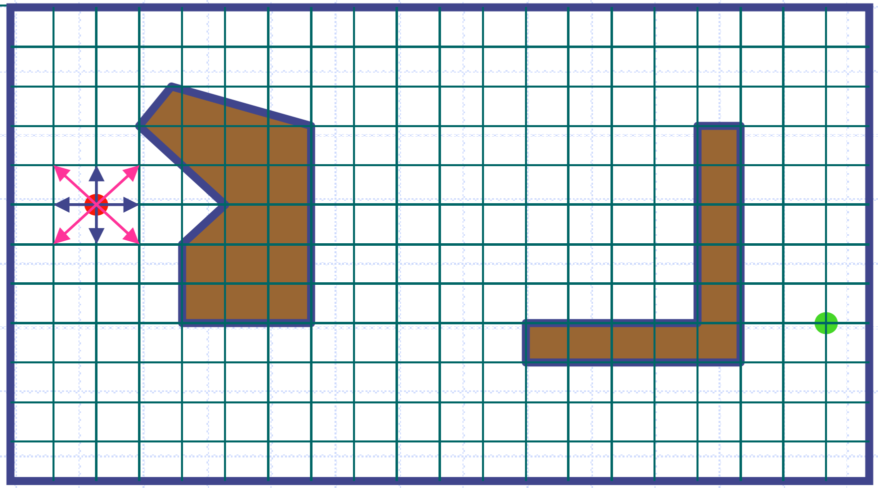
Robot Navigation Heuristics



Cost of one horizontal/vertical step = 1
Cost of one diagonal step = $\sqrt{2}$

$$h_1(N) = \sqrt{(x_N - x_g)^2 + (y_N - y_g)^2}$$ is ADMISSIBLE

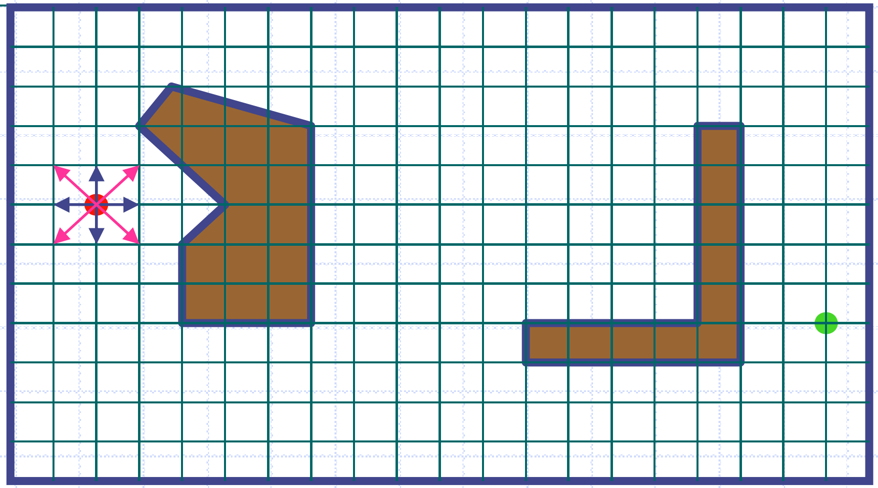
Robot Navigation Heuristics



Cost of one horizontal/vertical step = 1
Cost of one diagonal step = $\sqrt{2}$

$h_2(N) = |x_N - x_g| + |y_N - y_g|$ is ???

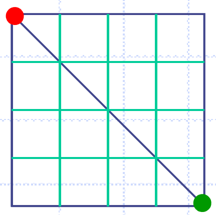
Robot Navigation Heuristics



Cost of one horizontal/vertical step = 1
Cost of one diagonal step = $\sqrt{2}$

$h_2(N) = |x_N - x_g| + |y_N - y_g|$ is admissible if moving along diagonals is not allowed, and not admissible otherwise

$h^*(I) = 4\sqrt{2}$
$h_2(I) = 8$



How to create admissible h?

- An admissible heuristic can usually be seen as the cost of an optimal solution to a **relaxed problem** (one obtained by removing constraints)
- In robot navigation:
  - The Manhattan distance corresponds to removing the obstacles
  - The Euclidean distance corresponds to removing both the obstacles and the constraint that the robot moves on a grid

A* Search (most popular algorithm in AI)

1. $f(N) = g(N) + h(N)$, where:
   - $g(N)$ = cost of best path found so far to N
   - $h(N)$ = admissible heuristic function
2. for all arcs: $c(N,N') \geq \varepsilon > 0$
3. SEARCH#2 algorithm is used

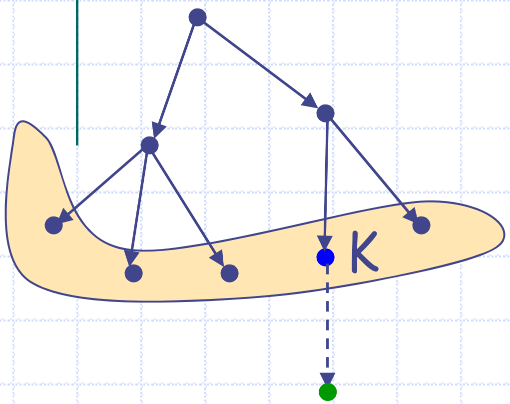
- Best-first search is then called **A* search**

Result #1

- A* is complete and optimal
  [This result holds if nodes revisiting states are not discarded]

Proof (1/2)

- If a solution exists, A* terminates and returns a solution

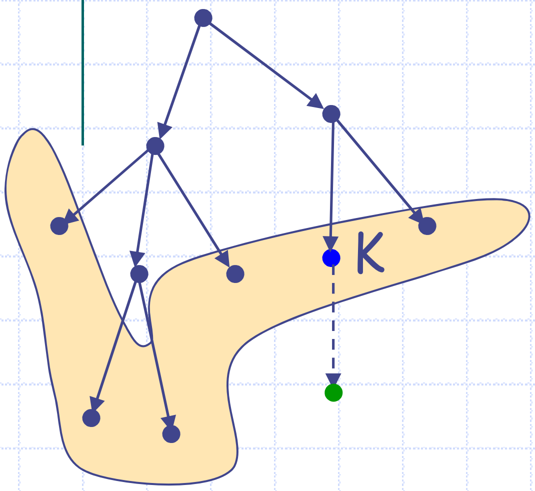


- For each node N on the fringe,
  $f(N) = g(N)+h(N) \geq g(N) \geq d(N)\times \varepsilon$,
  where d(N) is the depth of N in the tree

- As long as A* hasn't terminated, a node K on the fringe lies on a solution path

Proof (1/2)

- If a solution exists, A* terminates and returns a solution



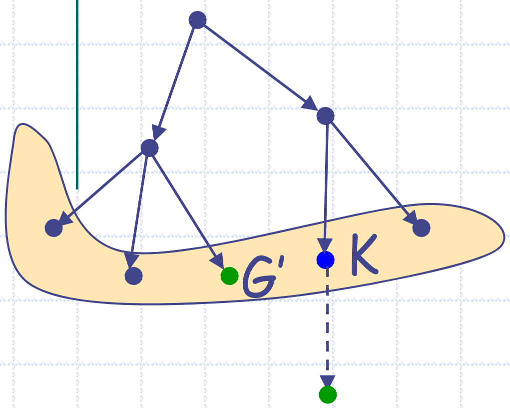
- For each node N on the fringe,
  $f(N) = g(N)+h(N) \geq g(N) \geq d(N) \times \varepsilon$,
  where d(N) is the depth of N in the tree

- As long as A* hasn't terminated, a node **K** on the fringe lies on a solution path

- Since each node expansion increases the length of one path, **K** will eventually be selected for expansion, unless a solution is found along another path

Proof (2/2)

- Whenever A* chooses to expand a goal node, the path to this node is optimal



- C*= h*(initial-node)
  [cost of the optimal solution path]

- G': non-optimal goal node in the fringe
  $f(G') = g(G') + h(G') = g(G') > C^*$

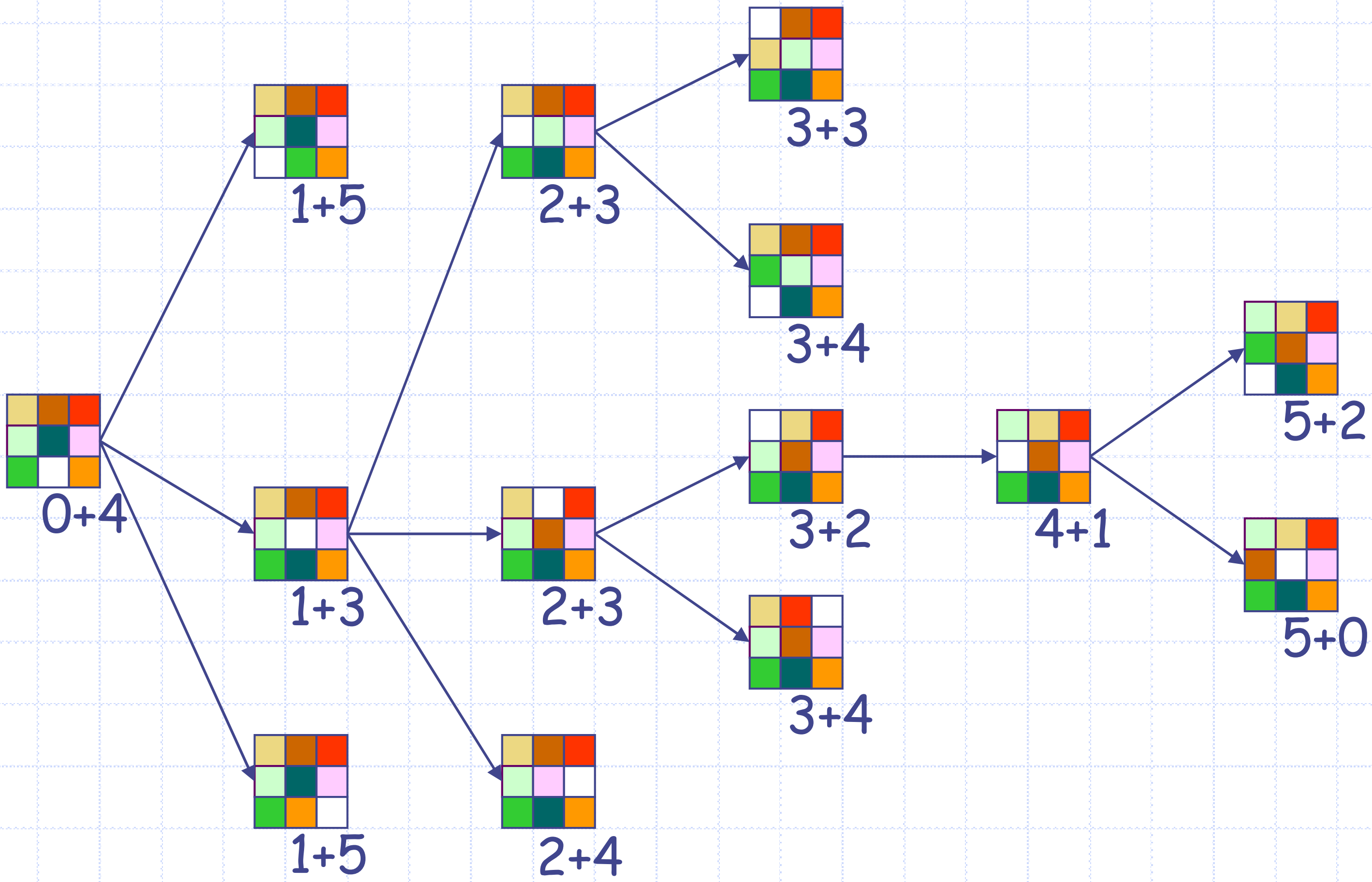
- A node **K** in the fringe lies on an optimal path:
  $f(K) = g(K) + h(K) \leq C^*$

  Optimistic estimate
  $h(N) \leq h^*(N)$

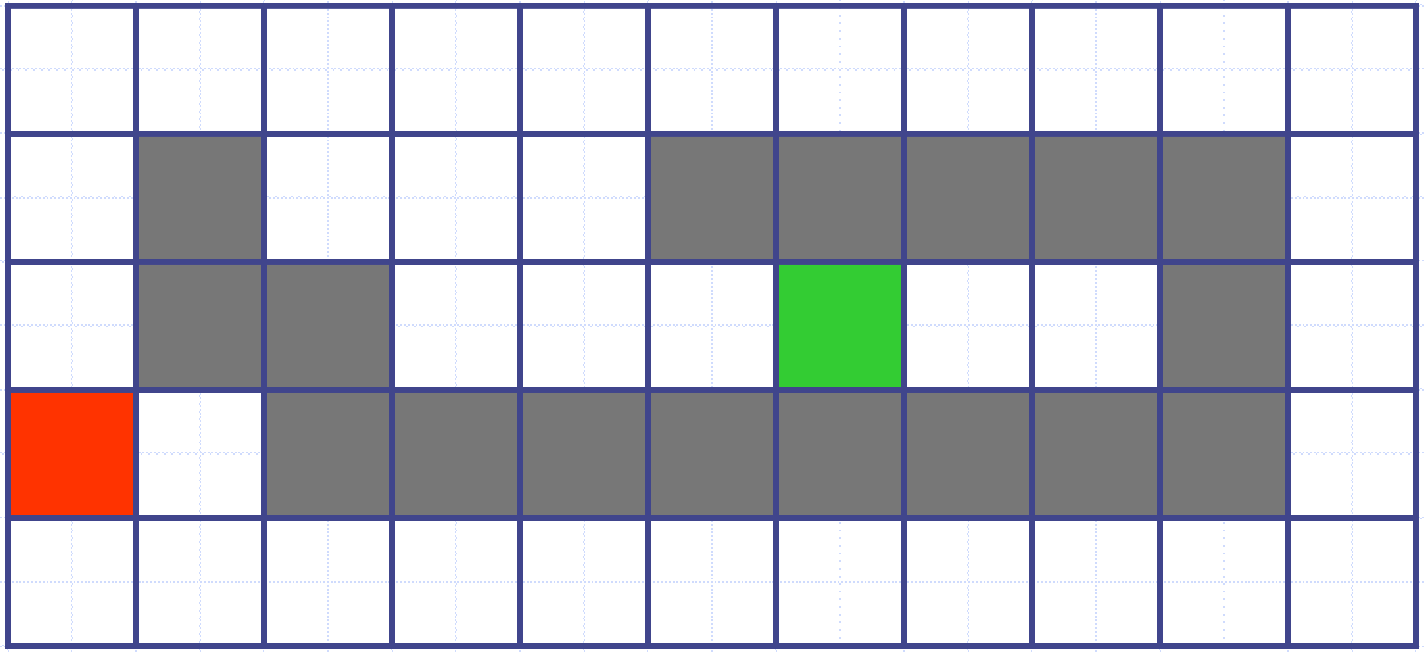
-So, G' will not be selected for expansion

8-Puzzle

$f(N) = g(N) + h(N)$
with $h(N)$ = number of misplaced tiles



Robot Navigation



Robot Navigation

f(N) = h(N), with h(N) = Manhattan distance to the goal (not A*)

8	7	6	5	4	3	2	3	4	5	6
7		5	4	3						5
6			3	2	1	0	1	2		4
7	6									5
8	7	6	5	4	3	2	3	4	5	6

Robot Navigation

f(N) = h(N), with h(N) = Manhattan distance to the goal
(not A*)

8	7	6	5	4	3	2	3	4	5	6
7		5	4	3						5
6			3	2	1	0	1	2		4
7	6									5
8	7	6	5	4	3	2	3	4	5	6

Robot Navigation

f(N) = g(N)+h(N), with h(N) = Manhattan distance to goal (A*)

8+3	7+4	6+3	5+6	4+7	3+8	2+9	3+10	4	5	6
7+2		5+6	4+7	3+8						5
6+1			3	2+9	1+10	0+11	1	2		4
7+0	6+1									5
8+1	7+2	6+3	5+4	4+5	3+6	2+7	3+8	4	5	6

Best-First Search

- An **evaluation function** f maps each node N of the search tree to a real number:

$$f(N) \geq 0$$

- **Best-first search** sorts the FRINGE in increasing f

A* Search (most popular algorithm in AI)

1. $f(N) = g(N) + h(N)$, where:
   - $g(N)$ = cost of best path found so far to N
   - $h(N)$ = admissible heuristic function
2. for all arcs: $c(N,N') \geq \varepsilon > 0$
3. SEARCH#2 algorithm is used

- Best-first search is then called **A* search**

Result #1

- A* is complete and optimal
  [This result holds if nodes revisiting states are not discarded]